Zero Energy States of Reduced Super Yang-Mills Theories in $d+1=4,6$ and 10 dimensions are necessarily Spin(d) invariant

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Abstract

We consider reduced Super Yang-Mills Theory in $d+1$ dimensions, where $d = 2,3,5,9$. We present commutators to prove that for $d = 3,5$ and 9 a possible ground state must be a Spin(d) singlet. We also discuss the case $d = 2$, where we give an upper bound on the total angular momentum and show that for odd dimensional gauge group no Spin(d) invariant state exists in the Hilbert space.

1 Introduction

We consider models, which are obtained by dimensional reduction of Super Yang-Mills theory with gauge group $SU(N)$ in $d+1$ dimensions, where $d = 2,3,5,9$. These models were used to formulate a quantum theory of supermembranes living in $d+2$ dimensions, and for $d = 9$ they describe N interacting $D0$ branes. It is interesting to know, whether these models admit a possible zero energy state and what the properties of such a state are. The general belief, partially proven, is that for $d = 2,3,5$ no zero energy state exists and that for $d = 9$ there exists a unique ground state.

Let us start with a very simple argument that zero energy states in $d = 9$ are Spin(d) invariant: it is well known [1, 2, 3] that the supercharges, Q_β, of reduced Yang-Mills theory (for definitions and conventions, see the next section) and

$$\widetilde{Q}_\beta = m\sum_{i=1}^{3} q_{iA}(\gamma^{123}\gamma^i)_{\beta\alpha}\Theta_{\alpha A} - \frac{m}{2}\sum_{\mu=4}^{9} q_{\mu A}(\gamma^{123}\gamma^\mu)_{\beta\alpha}\Theta_{\alpha A}\ ,$$

$\gamma^{123} = \gamma^1\gamma^2\gamma^3$, satisfy anti commutation relations of the form

$$\{Q_\beta + \widetilde{Q}_\beta, Q_{\beta'} + \widetilde{Q}_{\beta'}\} = \delta_{\beta\beta'}\widehat{H} + mJ_{ij}(\gamma^{123}\gamma^{ij})_{\beta\beta'} - \frac{m}{2}J_{\mu\nu}(\gamma^{123}\gamma^{\mu\nu})_{\beta\beta'} + 2q_{tA}\gamma^t_{\beta\beta'}J_A,$$

where $J_A, J_{ij}, J_{\mu\nu}$ are the generators of $SU(N)$, Spin(3), Spin(6) respectively and $\widehat{H}$ is an operator, whose form is not important here. As

$$\{\widetilde{Q}_\beta, \widetilde{Q}_{\beta'}\} = \delta_{\beta\beta'}(m^2 \sum_{iA} q_{iA}^2 + \frac{m^2}{4} \sum_{\mu A} q_{\mu A}^2) =: \left(\delta_{\beta\beta'} \widetilde{H}\right) ,$$

it immediately follows that

$$\begin{aligned} \{Q_\beta, \widetilde{Q}_{\beta'}\} + \{\widetilde{Q}_\beta, Q_{\beta'}\} &= \delta_{\beta\beta'}(\widehat{H} - H - \widetilde{H}) \\ &\quad + m J_{ij}(\gamma^{123}\gamma^{ij})_{\beta\beta'} - \frac{m}{2} J_{\mu\nu}(\gamma^{123}\gamma^{\mu\nu})_{\beta\beta'} , \end{aligned} \tag{1}$$

so that for $SU(N)$ invariant zero energy states ϕ, ψ, i.e. states annihilated by the Q_β and J_A,

$$(\phi, J_{ij}\psi) = 0, \ \ (\phi, J_{\mu\nu}\psi) = 0.$$

(just multiply (1) by $(\gamma^{123}\gamma^{ij})_{\beta\beta'}$, respectively $(\gamma^{123}\gamma^{\mu\nu})_{\beta\beta'}$ and sum over β and β'); hence

$$J_{ij}\psi = 0, \ \ J_{\mu\nu}\psi = 0,$$

by choosing $\phi = J_{ij}\psi$, respectively $J_{\mu\nu}\psi$. As (123) may be replaced by any other triple (stu), $J_{st}\psi = 0$, for all $s, t = 1, ...9$, provided $Q_\beta\psi = 0, J_A\psi = 0$.

In the next section we will treat $d = 2, 3, 5, 9$ on equal footing and, similar to [4], look for anti-commutators to prove that zero-energy states have to be invariant under Spin(d). We do find such anti-commutators for $d = 3, 5$ and 9. For $d = 2$, we give an upper bound on the total angular momentum and show that if $SU(N)$ is odd dimensional, i.e. N even, no Spin(d) invariant state exists in the Hilbert space. The discussion below generalizes to other gauge groups.

2 Model and Results

Let $d = 2, 3, 5, 9$, and let $(\gamma^i)_{\alpha\beta}$ denote the real irreducible representation of smallest dimension, called s_d, of the γ-matrices in d dimensions, i.e. the relations $\{\gamma^s, \gamma^t\} = 2\delta^{st}\mathbb{1}$. We have $s_d = 2, 4, 8, 16$. The model, which we are discussing, contains the self adjoint bosonic degrees of freedom q_{sA}, p_{sA} $(s = 1, ..., d,\ A = 1, ..., N^2 - 1)$ and the self adjoint fermionic degrees of freedom $\Theta_{\alpha A}$ $(\alpha = 1, ..., s_d,\ A = 1, ..., N^2 - 1)$ satisfying

$$[q_{sA}, p_{tB}] = i\delta_{st}\delta_{AB}, \tag{2}$$

$$\{\Theta_{\alpha A}, \Theta_{\beta B}\} = \delta_{\alpha\beta}\delta_{AB}, \tag{3}$$

$$[q_{sA}, \Theta_{\alpha B}] = [p_{sA}, \Theta_{\alpha B}] = 0.$$

More precisely, we consider the Schrödinger representation $(p_{sA} = -i\partial_{sA})$ of (2) on the Hilbert space

$$\mathcal{H} = L^2(\mathbb{R}^{d(N^2-1)}) \otimes \mathcal{F},$$

where $\mathcal{F} \cong (\mathbb{C}^2)^{(s_d/2)(N^2-1)}$ is the irreducible representation space of (3). The infinitesimal generators of the gauge group $SU(N)$ read

$$J_A = -if_{ABC}(q_{tB}\partial_{tC} + \frac{1}{2}\Theta_{\alpha B}\Theta_{\alpha C}),$$

where f_{ABC} are real, antisymmetric structure constants of $SU(N)$. The physical Hilbert space $\mathcal{H}_{\text{phys}}$, given by the $SU(N)$ invariant states in $\mathcal{H}$, is the Hilbert space of the model. We have a representation of $\text{Spin}(d)$ on $\mathcal{H}$ ($\mathcal{H}_{\text{phys}}$), with infinitesimal generators

$$\begin{aligned} J_{st} &= -i(q_{sA}\partial_{tA} - q_{tA}\partial_{sA}) - \frac{i}{4}\Theta_{\alpha A}\gamma^{st}_{\alpha\beta}\Theta_{\beta A} \\ &\equiv L_{st} + M_{st}\ , \end{aligned}$$

where $\gamma^{st} = \frac{1}{2}[\gamma^s, \gamma^t]$. The supercharges are given by

$$Q_\beta = \Theta_{\alpha A}(-i\gamma^t_{\alpha\beta}\partial_{tA} + \frac{1}{2}f_{ABC}q_{sB}q_{tC}\gamma^{st}_{\beta\alpha})\ ,$$

and the Hamiltonian by

$$H = -\Delta + \frac{1}{2}f_{ABC}q_{sB}q_{tC}f_{ADE}q_{sD}q_{tE} + iq_{sA}f_{ABC}\Theta_{\alpha B}\Theta_{\beta C}\gamma^s_{\alpha\beta}\ .$$

The anti-commutation relations for the supercharges are

$$\{Q_\alpha, Q_\beta\} = \delta_{\alpha\beta}H + 2\gamma^s_{\alpha\beta}q_{sA}J_A\ ,$$

On $\mathcal{H}_{\text{phys}}$ this reads

$$\{Q_\alpha, Q_\beta\} = \delta_{\alpha\beta}H.$$

We note that the Operators Q_α and H are self adjoint on their maximal domain, i.e.

$$\begin{aligned} \mathcal{D}(Q_\alpha) &= \{\psi \in \mathcal{H} | (Q_\alpha\psi)_{\text{dist}} \in \mathcal{H}\}, \\ \mathcal{D}(H) &= \{\psi \in \mathcal{H} | (H\psi)_{\text{dist}} \in \mathcal{H}\}, \end{aligned}$$

where $(\,\cdot\,)_{\text{dist}}$ is understood in the sense of distributions. The restrictions of H and Q_α to $\mathcal{H}_{\text{phys}}$ are also self adjoint. We are only interested in $SU(N)$ invariant states, i.e. states in $\mathcal{H}_{\text{phys}}$. By definition ψ is a zero energy state iff $\psi \in \mathcal{H}_{\text{phys}} \cap \text{Ker}H$. We want to prove the following

Theorem 1.

(a) *For $d = 3, 5, 9$, a possible zero energy state is a* Spin(d) *singlet.*

(b) *For $d = 2$, a possible zero energy state ψ satisfies*

$$\|J_{st}\psi\| \le \frac{3}{2}\cdot \dim SU(N)\, \|\psi\|.$$

We start with

Lemma 2. *We have the following (formal) anti-commutator relations.*

(a) *For $d = 2, 3, 5, 9$, we have with $b^{uv}_\alpha := \frac{1}{s_d}(q_{uA}\gamma^v_{\alpha\epsilon}\Theta_{\epsilon A} - q_{vA}\gamma^u_{\alpha\epsilon}\Theta_{\epsilon A})$,*

$$\{Q_\alpha, b^{uv}_\alpha\} = J_{uv} + (8/s_d - 1)M_{uv}\ .$$

(b) *For $d = 2, 3, 9$, we have with $c_\alpha^{uv} := \frac{1}{s_d}(\gamma^w\gamma^{uv})_{\alpha\epsilon}q_{wD}\Theta_{\epsilon D}$,*

$$\{Q_\alpha, c_\alpha^{uv}\} = J_{uv} + (4d/s_d - 1)M_{uv}$$

(c) *For $d = 3, 5, 9$, we have with $a_\alpha^{uv} = \frac{1}{4d-8} \cdot ((4d - s_d)b_\alpha^{uv} - (8 - s_d)c_\alpha^{uv})$,*

$$\{Q_\alpha, a_\alpha^{uv}\} = J_{uv}.$$

Proof.
By a straight forward calculation we find for $d = 2, 3, 5, 9$

$$\{Q_\alpha, q_{uD}\Theta_{\epsilon F}\} = -i\gamma^u_{\beta\alpha}\Theta_{\beta D}\Theta_{\epsilon F} + \gamma^s_{\epsilon\alpha}q_{uD}(-i\partial_{sF}) + \frac{1}{2}f_{FBC}q_{uD}q_{sB}q_{tC}\gamma^{st}_{\alpha\epsilon}\ . \tag{4}$$

(a) We have, using (4),

$$\{Q_\alpha, q_{uD}\gamma^v_{\alpha\epsilon}\Theta_{\epsilon D}\} = -i(\gamma^u\gamma^v)_{\beta\epsilon}\Theta_{\beta D}\Theta_{\epsilon D} + s_d q_{uD}(-i\partial_{vD}) + \frac{1}{2}f_{DBC}q_{uD}q_{sB}q_{tC}\gamma^{st}_{\alpha\epsilon}\gamma^v_{\alpha\epsilon}\ . \tag{5}$$

The last term in (5) vanishes since the trace over the γ-matrices equals zero. We find

$$\begin{aligned} \{Q_\alpha, q_{uA}\gamma^v_{\alpha\epsilon}\Theta_{\epsilon A} - q_{vA}\gamma^u_{\alpha\epsilon}\Theta_{\epsilon A}\} &= -is_d(q_{uA}\partial_{vA} - q_{vA}\partial_{uA}) - i\Theta_{\alpha A}2\gamma^{uv}_{\alpha\epsilon}\Theta_{\epsilon A} \\ &= s_d J_{uv} + (8 - s_d)M_{uv}\ . \end{aligned}$$

(b) We have, using (4),

$$\begin{aligned} \{Q_\alpha, (\gamma^w\gamma^{uv})_{\alpha\epsilon}q_{wD}\Theta_{\epsilon D}\} &= d(-i)\gamma^{uv}_{\beta\epsilon}\Theta_{\beta D}\Theta_{\epsilon D} + s_d(-i)(q_{uD}\partial_{vD} - q_{vD}\partial_{uD}) \\ &\quad -\frac{1}{2}\mathrm{Tr}(\gamma^w\gamma^{uv}\gamma^{st})f_{DBC}q_{wD}q_{sB}q_{tC} \\ &= s_d J_{uv} + (4d - s_d)M_{uv}\ , \end{aligned}$$

where the term in the second line is zero, as the trace over the five γ-matrices vanishes.
(c) follows by a linear combination of (a) and (b). □

We note that the action of Spin(d) leaves the kernel of H invariant. Let $\varphi, \psi \in \mathrm{Ker}H \cap \mathcal{H}_{\mathrm{phys}}$. Then $\varphi, \psi \in \mathrm{Ker}Q_\beta$ for all β and by elliptic regularity $\varphi, \psi \in C^\infty$. We assume that ψ lies in an irreducible representation space of Spin(d). Hence $J_{uv}\psi \in \mathrm{Ker}H \cap \mathcal{H}_{\mathrm{phys}}$. Let $d = 3, 5, 9$. By Lemma 2 (c), we have

$$Q_\alpha a_\alpha^{uv}\psi = \{Q_\alpha, a_\alpha^{uv}\}\psi = J_{uv}\psi\ \ \in \mathcal{H}\ .$$

Taking the scalar product with φ, we want to bring Q_α to the other side, i.e. integrate by parts. Therefore we regularize as in [4]. There exists a function $\chi : [0, \infty) \to \mathbb{R}$ in C^∞, such that

$$\chi(r) = \begin{cases} 1 & r \le 1 \\ \in [0,1] & 1 < r < 3 \\ 0 & 3 \le r \end{cases}\ ,$$

and $|\chi'(r)| \leq 1$. Define $g_n(q) \equiv \chi(|q|/n)$. By dominated convergence,

$$\begin{aligned}(\varphi, J_{uv}\psi) &= \lim_{n\to\infty} (\varphi, g_n Q_\alpha a_\alpha^{uv}\psi) \\ &= \lim_{n\to\infty} (\varphi, [g_n, Q_\alpha] a_\alpha^{uv}\psi) + \lim_{n\to\infty} (\varphi, Q_\alpha g_n a_\alpha^{uv}\psi)\ . \qquad (6)\end{aligned}$$

The second term in (6) vanishes since $g_n a_\alpha^{uv}\psi \in C_0^\infty$ is in the domain of Q_α and Q_α is self adjoint. By

$$|[Q_\beta, g_n]|_{\mathcal{F}} \leq \text{const.} \cdot \frac{1}{n}\chi'(|q|/n),$$

where $|\cdot|_{\mathcal{F}}$ stands for the the norm in $\mathcal{F}$ or the operator norm in $\mathcal{L}(\mathcal{F})$, the first term in (6) vanishes using the following estimate.

$$|(\varphi, [g_n, Q_\alpha] a_\alpha^{uv}\psi)| \leq \text{const.} \cdot \int_{n\leq|q|\leq 3n} n \cdot \frac{1}{n} |\varphi|_{\mathcal{F}} |\psi|_{\mathcal{F}} \mathrm{d}q \ \to 0 \qquad \text{for} \ \ n \to \infty.$$

Hence $(\varphi, J_{uv}\psi) = 0$. Choosing $\varphi = J_{uv}\psi$, we find

$$(J_{uv}\psi, J_{uv}\psi) = 0.$$

By linear combination, it follows that for $d = 3, 5, 9$ all states in $\mathrm{Ker} H \cap \mathcal{H}_{\text{phys}}$ are $\mathrm{Spin}(d)$ singlets. For $d = 2$, we use Lemma 2 (a) or (b) and find by an analogous argument $(\varphi, (J_{st} + 6M_{st})\psi) = 0$. Choosing $\varphi = J_{st}\psi$, we obtain

$$\begin{aligned}(J_{12}\psi, J_{12}\psi) &\leq 6|(J_{12}\psi, M_{12}\psi)| \\ &\leq 6\|J_{12}\psi\| \cdot \|M_{12}\psi\|\ .\end{aligned}$$

A real irreducible representation of the γ-matrices in 2 dimensions is given by $\gamma^1 = \sigma^1$, $\gamma^2 = -\sigma^3$. In this representation we have $\gamma^{12} = \frac{1}{2}[\gamma^1, \gamma^2] = i\sigma^2$. It follows that

$$\begin{aligned}\|J_{12}\psi\| &\leq 6\|M_{12}\psi\| \\ &= 6\|\frac{i}{4}\Theta_{\alpha A}\gamma^{12}_{\alpha\beta}\Theta_{\beta A}\psi\| \\ &= 6\|\frac{i}{2}\Theta_{1A}\Theta_{2A}\psi\| \\ &\leq \frac{3}{2}\dim SU(N)\|\psi\|.\end{aligned}$$

By linear combination the above equation holds for all states in $\mathrm{Ker} H \cap \mathcal{H}_{\text{phys}}$. Hence Theorem 1 follows.

The case $d = 2$ is special as the following theorem shows.

Theorem 3. *For $d = 2$ and odd dimensional gauge group $SU(N)$ no* $\mathrm{Spin}(d)$ *invariant state exists in $\mathcal{H}$.*

Proof. By definition

$$J_{12} = -i(q_{1A}\partial_{2A} - q_{2A}\partial_{1A}) - \frac{i}{4}\Theta_{\alpha A}\gamma^{12}_{\alpha\beta}\Theta_{\beta A}.$$

As above, we choose $\gamma^1 = \sigma^1$, $\gamma^2 = -\sigma^3$. We define the following annihilation and creation operators

$$\frac{\partial}{\partial \lambda_A} = \frac{1}{\sqrt{2}}(\Theta_{1A} + i\Theta_{2A}), \qquad \lambda_A = \frac{1}{\sqrt{2}}(\Theta_{1A} - i\Theta_{2A}).$$

We find

$$J_{12} = L_{12} - \frac{i}{2}\Theta_{1A}\Theta_{2A} = L_{12} - \frac{1}{2}\lambda_A \frac{\partial}{\partial \lambda_A} + \frac{1}{4} \cdot \dim SU(N).$$

Assume ψ is Spin(d)-invariant, i.e. $J_{12}\psi = 0$. Then

$$\left(L_{12} - \frac{1}{2}\lambda_A \frac{\partial}{\partial \lambda_A} \right) \psi = -\frac{1}{4} \cdot \dim SU(N) \psi.$$

If $\dim SU(N)$ is odd this contradicts that the spectrum of $L_{12} - \frac{1}{2}\lambda_A \frac{\partial}{\partial \lambda_A}$ only takes values in $\frac{1}{2}\mathbb{Z}$. Hence the claim follows. $\square$

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